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The hierarchy of multi-soliton solutions of the derivative nonlinear Schrödinger equation

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Abstract
We provide a relatively simple approach to Bäcklund transformations for the derivative nonlinear Schrödinger equation. By iteration it leads to compact N-soliton formulae both with asymptotically vanishing and non-vanishing amplitudes. The phenomenology of these solutions is discussed and illustrated in some detail.

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1. Introduction

The *derivative nonlinear Schrödinger* (DNLS) *equation*,

$$q_t + \mathrm{i}q_{xx} + (q^2 q^*)_x = 0 \tag{1}$$

has attracted considerable attention both from the theoretical point of view and with respect to physical applications. Here the star means complex conjugation, and subscripts x, t denote derivatives.

In plasma physics it has long been known that the DNLS equation governs the evolution of small but finite amplitude Alfvén waves propagating quasi-parallel to the magnetic field in a low-β plasma [1–4], β being the ratio of kinetic to magnetic pressure. Recently it was shown [5] that the same equation describes the behaviour of large-amplitude magnetohydrodynamic waves propagating in an arbitrary direction with respect to the magnetic field in a high-β plasma as well. Further, the filamentation of lower-hybrid waves can be modelled by the DNLS equation [6], and dark DNLS solitons have been proposed for the interpretation of 'magnetic holes' in space plasmas [7].

In nonlinear optics the propagation of light pulses in an optical waveguide is described by the usual (cubic) *nonlinear Schrödinger* (NLS) *equation*. For very short pulses the typical Kerr nonlinearity has to be supplemented by a derivative term [8, 9]. As was first shown by Ichikawa *et al* [10] that the NLS equation generalized in such a way may be transformed to the DNLS equation.

Integrability in the sense of the inverse scattering method was shown for the DNLS equation by Kaup and Newell [11]. In the same paper integral equations of Gelfand–Levitan–Marchenko (GLM) type were established and used to get a one-soliton solution over vacuum.

The first N-soliton formula for the DNLS equation was established by Nakamura and Chen [12] by use of Hirota's bilinear transform method. On the basis of Darboux transformation technique, Huang and Chen [13] derived an N-soliton formula in terms of determinants. A related comment by Xiao [14] stresses on a parallel treatment of Darboux transformations for NLS and DNLS equations. All solutions given in the above-quoted papers are asymptotically vanishing.

Kawata and Inoue [15] succeeded in applying the inverse scattering technique to the problem with a finite-asymptotic wave, to get the related GLM equations and to derive bright and dark solitons as well as breather-type solitons called *paired solitons* by these authors. This procedure, however, is rather laborious and does not seem well suited for establishing convenient N-soliton formulae. Eichhorn [16] derived a Bäcklund transformation for the generalized NLS equation with the usual non-derivative third-order term besides the derivative term, and arrived at an N-soliton formula in determinant form. An interesting method to get Bäcklund transformations from gauge transformations was presented by Kundu [17] and applied, in particular, to the DNLS equation. It seems, however, not easy to get explicit formulae for iterations.

Kamchatnov developed a method for finding periodic solutions of several integrable evolution equations and applied this method to the DNLS equation [18–20]. One-soliton solutions with zero or nonzero asymptotics are found as limiting cases. The Cauchy problem for the DNLS equation has been discussed by Hayashi and Ozawa [21]. The formation of solitons on the sharp front of an optical pulse in a fibre was treated on the basis of the DNLS equation by Kamchatnov *et al* [22].

In the present paper we offer a relatively simple and elementary approach to Darboux/Bäcklund transformations which, by iteration, leads to N-soliton formulae of a rather transparent structure, and we demonstrate that these formulae are well suited for generating computer pictures of N-soliton states up to $N = 8$, at least. The principal features of our procedure have been outlined already in [23]. In a recent paper [24] the same method has been applied to second-harmonic generation with account of the Kerr effect.

Our method works for a finite background as well as with vacuum asymptotics. We make use of *Vandermonde-type determinants* [25] and introduce what we call the *seahorse function*. These notions are defined in appendices A and B, respectively.

In section 2 from a system of two differential equations as a generalization of (1) we list some useful symmetries and write down the crucial simultaneous linear system as well as the corresponding Riccati equations. In section 3 the Bäcklund transformation for the DNLS equation is written down, and N-soliton formulae are given. Particular soliton solutions both over vacuum and over finite background are discussed in some detail in section 4. A short discussion on stability is given in section 5 followed by a summary and conclusions in section 6 where also a comparison with a related treatment of Darboux transformations and DNLS solitons by Imai [26] is given.

2. Basic equations and symmetries

Let us start from a system of two differential equations,

$$-\mathrm{i}q_t + q_{xx} - \mathrm{i}(q^2 r)_x = 0 \tag{2}$$

$$\mathrm{i}r_t + r_{xx} + \mathrm{i}(r^2 q)_x = 0 \tag{3}$$

which reduces to the DNLS equation (1) for $r = q^*$ while the choice $r = -q^*$ would lead to (1) with the sign of the nonlinear term changed. There is some advantage to treat q and r as mutually independent and to require $r = \pm q^*$ afterward by *reduction*.

The system (2), (3) is invariant under the following transformations:

1. Space reflection,
$$\tilde{x} = -x \qquad \tilde{t} = t \qquad \tilde{q} = q \qquad \tilde{r} = -r. \tag{4}$$
2. Time reflection,
$$\tilde{x} = x \qquad \tilde{t} = -t \qquad \tilde{q} = r \qquad \tilde{r} = -q. \tag{5}$$
3. Conjugation,
$$\tilde{q} = r^* \qquad \tilde{r} = q^*. \tag{6}$$
4. Scale transformation,
$$\tilde{x} = bx \qquad \tilde{t} = b^2 t \qquad \tilde{q} = q/b \qquad \tilde{r} = r/b. \tag{7}$$
5. Gauge transformation,
$$\tilde{q} = q\,\mathrm{e}^{\mathrm{i}\kappa} \tag{8}$$
with κ being an arbitrary real constant.

Due to symmetries 1 and 2 the reductions $r = q^*$ and $r = -q^*$ are interchanged by space reflection as well as by time reflection. Thus it is enough to discuss one of these two cases, and we prefer $r = q^*$ to get the DNLS equation in the form (1). Symmetry 4 permits us to normalize the amplitude of a monochromatic wave, and symmetry 5 permits us to ignore a constant phase.

Equations (2) and (3) are the compatibility conditions of the linear system [11]

$$\partial_x \phi = (J\zeta^2 + Q\zeta)\phi \equiv U\phi \tag{9}$$

$$\partial_t \phi = (-2J\zeta^4 + V_3\zeta^3 + V_2\zeta^2 + V_1\zeta)\phi \equiv V\phi \tag{10}$$

with

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad J = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{11}$$

$$V_3 = -2Q \qquad V_2 = -Jqr \qquad V_1 = \begin{pmatrix} 0 & -\mathrm{i}q_x - q^2 r \\ \mathrm{i}r_x - r^2 q & 0 \end{pmatrix}. \tag{12}$$

Here ζ is an arbitrary complex number called the *spectral parameter*. Equations (2) and (3) are equivalent to the integrability condition $U_t - V_x + [U, V] = 0$ of (9) and (10).

A system characterized by (9) is called a *quadratic bundle*. Gerdzhikov *et al* [27, 28] have analysed the mathematical structure of such systems and their connection to completely integrable Hamiltonian systems including, in particular, the DNLS equation.

The system (9), (10) is covariant under the mapping

$$\tilde{\varphi}_1 = \varphi_1 \qquad \tilde{\varphi}_2 = -\varphi_2 \qquad \tilde{\zeta} = -\zeta. \tag{13}$$

It is easy to see that the quotient

$$\beta \equiv \frac{\varphi_2}{\varphi_1} \tag{14}$$

fulfils the simultaneous Riccati equations

$$\beta_x = \zeta(r + 2\mathrm{i}\zeta\beta - q\beta^2) \tag{15}$$

$$\beta_t = \zeta(-2\zeta^2 r + \mathrm{i}r_x - r^2 q) - 2\mathrm{i}\zeta^2(2\zeta^2 + qr)\beta + \zeta(2\zeta^2 q + \mathrm{i}q_x + q^2 r)\beta^2. \tag{16}$$

The linear system (9), (10) can be replaced by the Riccati system (15), (16) in the sense that (2) and (3) are the integrability conditions for this Riccati system as well.

It is useful to note that in the case of $r = q^*$ equations (9) and (10) are invariant under the mapping

$$\tilde{\varphi}_1 = \varphi_2^* \qquad \tilde{\varphi}_2 = \varphi_1^* \qquad \tilde{\zeta} = \zeta^* \tag{17}$$

and that, correspondingly, from β being a solution to (15) and (16) it follows that $1/\beta^*$ solves the same equations with ζ replaced by ζ^*.

For real ζ it follows from (15) and (16)

$$\partial_x(\beta^*\beta) = (\cdots) \times (\beta^*\beta - 1) \qquad \partial_t(\beta^*\beta) = (\cdots) \times (\beta^*\beta - 1) \tag{18}$$

so that if $|\beta| = 1$ holds anywhere the same holds everywhere. The details of the multiplicative factors $(\cdots)$ are not of interest for our conclusion.

3. Darboux/Bäcklund transformations

Comment concerning the notations. The term *Darboux transformation* denotes a method to derive from one solution of some scattering problem—here given by (9)—with specified *potentials* q, r a new solution with transformed potentials. When extended to a simultaneous linear system—here (9) and (10)—Darboux transformations become *Bäcklund transformations* which then include a transformation of solutions of the related nonlinear evolution equation(s)—(2) and (3) in the present case.

Iteration of this procedure leads to a hierarchy of solutions. When starting from trivial *vacuum* solutions typically one either arrives at *solitons* or—if the respective evolution equation does not admit solitons—one gets singular solutions. With respect to our present example, however, we shall see that Bäcklund transformations applied to vacuum generate particular monochromatic waves.

When we start from some solution with an asymptotically non-vanishing potential then, depending on the choice of parameters, we arrive at *solitons over a finite background* or at periodic solutions.

We quote a short list of literature on Darboux/Bäcklund transformations [29–34].

3.1. One single transformation

Now we are considering a spectral problem (9),

$$\phi_x = \begin{pmatrix} -\mathrm{i}\zeta^2 & \zeta q(x) \\ \zeta r(x) & \mathrm{i}\zeta^2 \end{pmatrix} \phi \equiv U\phi \tag{19}$$

and we repeat the approach to Darboux transformations as established in [23]. The quotient $\beta \equiv \varphi_2/\varphi_1$ solves the Riccati equation

$$\beta_x = \zeta(r + 2\mathrm{i}\zeta\beta - q\beta^2). \tag{20}$$

In order to establish a Darboux transformation we assume that one particular solution $\{\beta_1(x), \zeta_1, q(x), r(x)\}$ to (20) is known, and from this solution we define the matrix

$$M = \begin{pmatrix} \zeta\beta_1 & -\zeta_1 \\ -\zeta_1 & \zeta\alpha_1 \end{pmatrix} \qquad \alpha_1 \equiv 1/\beta_1. \tag{21}$$

Theorem. *From any solution* $\{\phi, \zeta, q, r\}$ *to (9) a new solution* $\{\tilde{\phi}, \zeta, \tilde{q}, \tilde{r}\}$ *is found by the Darboux transformation*

$$\tilde{\phi} = M\phi \qquad \tilde{q} = \beta_1(\beta_1 q - 2\mathrm{i}\zeta_1) \qquad \tilde{r} = \alpha_1(\alpha_1 r + 2\mathrm{i}\zeta_1). \tag{22}$$

The proof is straightforward by direct verification.

For later use it is important to note that $M\phi = 0$ for $\zeta = \zeta_1$.

Reduction. If $r = q^*$, ζ_1 real and β_1 is chosen to be of modulus 1—cf the remark at the end of section 3—then it follows $\alpha_1 = \beta_1^*$, $\tilde{r} = \tilde{q}^*$. That is, the symmetry $r = q^*$ is conserved.

Bäcklund transformation. It can be confirmed in a straightforward way that the matrix function V is form invariant under the transformation (22), i.e. $\tilde{V} \equiv (M_t + MV)M^{-1}$ is the same as V but with q, r replaced by $\tilde{q}, \tilde{r}$.

Commutativity. It is important to verify that two Darboux transformations commute. This can be done in a direct and straightforward way.

3.2. The N-fold Bäcklund transform

Now we return to the spectral problem (9) (with no reduction so far). Given one solution $q(x,t), r(x,t)$ to (2) and (3) and N solutions $\{q, r, \beta_k, \zeta_k\}$, $k = 1, \ldots, N$, to (15) then for the N-fold Darboux transform the wavefunction is an Nth-order polynomial in ζ,

$$\phi^{[N]} = \sum_{k=0}^{N} P_k \zeta^{N-k} \phi. \tag{23}$$

Below the theorem in section 3.1 it was stated that $M\phi = 0$ for $\zeta \to \zeta_1$. Together with commutativity now it follows that

$$\sum_{k=0}^{N} P_k \zeta_j^{N-k} \phi_j = 0. \tag{24}$$

From the iteration of (21) and (22) the coefficients P_k get the structure

$$P_{2l-1} = \begin{pmatrix} 0 & p_{2l-1} \\ s_{2l-1} & 0 \end{pmatrix} \qquad P_{2l} = \begin{pmatrix} p_{2l} & 0 \\ 0 & s_{2l} \end{pmatrix} \qquad p_N = s_N = -1 \tag{25}$$

where we took the value -1 in the last equation using the freedom of an arbitrary constant overall factor.

Equation (24) decays into two separate systems of linear equations for the coefficients p_k and s_k, respectively, and these two systems may be solved according to Cramer's rule. We have to distinguish whether N is odd or even. For later use we will write down the coefficients p_0 and p_1 explicitly, and we will use the notation of Vandermonde-like determinants, see appendix A.

(i) N odd, $N = 2n + 1$,

$$\sum_{l=1}^{n} p_{2l-1} \zeta_j^{2(n-l+1)} + \sum_{l=0}^{n} p_{2l} \zeta_j^{2n-2l+1} \alpha_j = 1 \tag{26}$$

$$\sum_{l=1}^{n} s_{2l-1} \zeta_j^{2(n-l+1)} + \sum_{l=0}^{n} s_{2l} \zeta_j^{2n-2l+1} \beta_j = 1 \tag{27}$$

$$p_0 = \frac{\mathcal{V}_{n+1,n}\left(1;\, \zeta_j\alpha_j \middle| \zeta_j^2\right)}{\mathcal{V}_{n,n+1}\left(\zeta_j^2;\, \zeta_j\alpha_j \middle| \zeta_j^2\right)} \tag{28}$$

$$p_1 = (-1)^{n-1}\frac{\mathcal{V}_{n,n+1}\left(1;\, \zeta_j\alpha_j \middle| \zeta_j^2\right)}{\mathcal{V}_{n,n+1}\left(\zeta_j^2;\, \zeta_j\alpha_j \middle| \zeta_j^2\right)}. \tag{29}$$

(ii) N even, $N = 2n$,

$$\sum_{l=0}^{n-1} p_{2l}\zeta_j^{2(n-l)} + \sum_{l=1}^{n} p_{2l-1}\zeta_j^{2(n-l)+1}\beta_j = 1 \tag{30}$$

$$\sum_{l=0}^{n-1} s_{2l}\zeta_j^{2(n-l)} + \sum_{l=1}^{n} s_{2l-1}\zeta_j^{2(n-l)+1}\alpha_j = 1 \tag{31}$$

$$p_0 = (-1)^{n-1}\frac{\mathcal{V}_{nn}\left(1;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2;\, \zeta_j\beta_j \middle| \zeta_j^2\right)} \tag{32}$$

$$p_1 = -\frac{\mathcal{V}_{n+1,n-1}\left(1;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2;\, \zeta_j\beta_j \middle| \zeta_j^2\right)}. \tag{33}$$

In both cases—N odd or even—s_0 and s_1 are easily found from the respective formulae for p_0 and p_1 because it holds generally

$$s_k = p_k(\alpha \longleftrightarrow \beta). \tag{34}$$

Let us write the transformed spectral problem in the form

$$\phi_x^{[N]} = \begin{pmatrix} -\mathrm{i}\zeta^2 & \zeta q^{[N]}(x) \\ \zeta r^{[N]}(x) & \mathrm{i}\zeta^2 \end{pmatrix}\phi \equiv U^{[N]}\phi. \tag{35}$$

Substitution of (23) into (9) and (35) and comparison of powers in ζ lead to

$$\begin{aligned}
&[P_0, J] = 0 \\
&[P_1, J] + P_0 Q - Q^{[N]} P_0 = 0 \\
&P_{k-1,x} + [P_{k+1}, J] + P_k Q - Q^{[N]} P_k = 0 \qquad k = 1, \ldots, N-1 \\
&P_{N-1,x} + P_N Q - Q^{[N]} P_N = 0.
\end{aligned} \tag{36}$$

and from the second equation of this system we get

$$q^{[N]} = \frac{p_0 q + 2\mathrm{i}p_1}{s_0} \qquad r^{[N]} = \frac{s_0 r - 2\mathrm{i}s_1}{p_0}. \tag{37}$$

When we choose the reduction $r = q^*$ we have to take the eigenvalues as real or as pairs of complex conjugate values and to choose

(i) $|\beta_j| = 1$ for real ζ_j or
(ii) $\beta_l = 1/\beta_k^* = \alpha_k^*$ when $\zeta_l = \zeta_k^*$.

Then we get $s_0 = p_0^*$, $s_1 = p_1^*$. Consequently, the required symmetry is conserved. The above formulae determine the N-fold *Darboux* transformation. When we 'switch on' the time t we know from subsection 4.1 that each of the N single transformation steps becomes a Bäcklund transformation, i.e. it transforms the simultaneous system (9), (10). Consequently, our result gives the N-fold *Bäcklund* transformation as well.

4. Particular solutions

4.1. Bäcklund transformations applied to the vacuum

For $q = r = 0$ the Riccati equations (15) and (16) are solved by

$$\beta = C\,\mathrm{e}^{2\mathrm{i}\zeta^2(x-2\zeta^2 t)} \tag{38}$$

with C being an arbitrary integration constant.

Case 1. $N = 1$. For one single Bäcklund transformation—due to the required reduction $r = q^*$, cf subsection 4.1—we have to take $\zeta = \zeta_1$ real and C of modulus 1. Ignoring an arbitrary phase we take $C = 1$,

$$\beta_1 = \mathrm{e}^{2\mathrm{i}\zeta_1^2(x-2\zeta_1^2 t)}. \tag{39}$$

The result of the Bäcklund transformation is then simply found from (22),

$$q^{[1]} = -2\mathrm{i}\zeta_1\beta_1. \tag{40}$$

This, of course, is not a soliton but a monochromatic wave. It is more special compared to the monochromatic wave given below, see (49) and (50). Principally, we could stop here and go to the next subsection where Bäcklund transformations are applied to the general monochromatic wave. On the other hand, however, it is worth to continue and find formulae which are hidden as limiting cases in more general manifolds of solutions given later.

Case 2. $N = 2$. We have to distinguish real eigenvalues and complex conjugate eigenvalues. But we may combine both cases by taking

$$\zeta_1 = l + m \qquad \zeta_2 = l - m \tag{41}$$

$$\text{(2a) } l, m \text{ real} \qquad \text{or} \qquad \text{(2b) } l = \xi \quad m = \mathrm{i}\eta \quad \xi \text{ and } \eta \text{ real.} \tag{42}$$

By specification of (32), (33) and (37) to $n = 1$ we find

$$q^{[2]} = -2\mathrm{i}\frac{\left(\zeta_1^2 - \zeta_2^2\right)(\zeta_1\beta_1 - \zeta_2\beta_2)}{(\zeta_1\beta_2 - \zeta_2\beta_1)^2}. \tag{43}$$

Now let us write $2\zeta_{1,2}^2\left(x - 2\zeta_{1,2}^2 t\right) = F \pm G$, i.e.

$$\begin{aligned} F &= 2(l^2 + m^2)x - 4(l^4 + m^4 + 6l^2m^2)t \\ G &= 4lm(x - 4(l^2 + m^2)t). \end{aligned} \tag{44}$$

Then (43) takes the form

$$\begin{aligned} q^{[2]} &= -4\mathrm{i}lm\,\mathrm{e}^{-\mathrm{i}F}\frac{m\cos G + \mathrm{i}l\sin G}{(m\cos G - \mathrm{i}l\sin G)^2} \\ &= -4\mathrm{i}lm\,\mathrm{e}^{-\mathrm{i}F}\frac{(m\cos G + \mathrm{i}l\sin G)^3}{(m^2 + (l^2 - m^2)\sin^2 G)^2}. \end{aligned} \tag{45}$$

This formula holds in both cases (2a) and (2b). For (2a) l, m, F, G are real. For (2b) F again is real while G is purely imaginary, $G = \mathrm{i}\Gamma$. Thus (45) in the latter case reads

$$\begin{aligned} q^{[2]} &= -4\mathrm{i}\xi\eta\,\mathrm{e}^{-\mathrm{i}F}\frac{\eta\cosh\Gamma + \mathrm{i}\xi\sinh\Gamma}{(\eta\cosh\Gamma - \mathrm{i}\xi\sinh\Gamma)^2} \\ &= -4\mathrm{i}\xi\eta\,\mathrm{e}^{-\mathrm{i}F}\frac{(\eta\cosh\Gamma + \mathrm{i}\xi\sinh\Gamma)^3}{(\eta^2 + (\xi^2 + \eta^2)\sinh^2\Gamma)^2}. \end{aligned} \tag{46}$$

This is a localized solution of stationary shape. We state that such a soliton over vacuum is generated by a two-fold Bäcklund transformation. It was first found by Mjølhus [2] and then recovered by Kaup and Newell [11] in the frame of the inverse scattering formalism.

Case 3. N-phase solution and n-soliton solution. Let us choose $N = 2n$ real eigenvalues ζ_j and define

$$\beta_j = \exp\left[2\mathrm{i}\zeta_j^2\left(x - 2\zeta_j^2 t\right)\right]. \tag{47}$$

Then from (32)–(34) and (37) we find

$$q^{[2n]} = (-1)^{n-1}\frac{\mathcal{V}_{n+1,n-1}\left(1;\, \zeta_j\beta_j\middle|\zeta_j^2\right)\mathcal{V}_{nn}\left(1;\, \zeta_j\beta_j\middle|\zeta_j^2\right)}{\left[\mathcal{V}_{nn}\left(\zeta_j;\, \beta_j\middle|\zeta_j^2\right)\right]^2}. \tag{48}$$

That is, the solution is a rational function of the exponentials β_j, $j = 1, \ldots, 2n$. (Note that $\alpha_j \equiv 1/\beta_j$.) Specified to $n = 1$ this solution coincides with (45), l and m real. Also we may take n pairs of complex conjugate eigenvalues ζ_j. Then by (48) we get an n-soliton solution. Of course, we might also choose M real eigenvalues together with m pairs of complex conjugate ones to get an m-soliton solution over an M-phase solution.

4.2. Bäcklund transformations applied to a monochromatic wave

It is easy to see that the monochromatic wave

$$q = a\,\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)} \qquad r = q^* \qquad a \text{ and } k \text{ real} \tag{49}$$

fulfils (2) and (3) provided the *nonlinear dispersion relation*

$$\omega_0 = k_0(a^2 - k_0) \tag{50}$$

is fulfilled. The more particular monochromatic wave (40) is characterized by $a^2 = 2k_0$.

We wish to solve (15) and (16) with q given by (49). If we write

$$\beta = -\mathrm{i}\gamma \exp[\mathrm{i}k_0(-x + (a^2 - k_0)t)] \tag{51}$$

we arrive at a simultaneous Riccati system with constant coefficients,

$$\mathrm{i}\partial_x\gamma = -2w\gamma - \zeta a(1+\gamma^2) \qquad w \equiv \zeta^2 + k_0/2 \tag{52}$$

$$\mathrm{i}\partial_t\gamma = (2\zeta^2 + a^2 - k_0)[2w\gamma + \zeta a(1+\gamma^2)]. \tag{53}$$

Its general solution may be written in terms of the so-called *seahorse function* as it is defined in three variants $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$ in appendix B,

$$\gamma = \mathcal{S}(A, dy/2) = \mathcal{S}_1(B, dy) = \mathcal{S}_2(\kappa, dy). \tag{54}$$

Here we have used the abbreviations

$$A = \frac{-w + \mathrm{i}d}{\zeta a} = \mathrm{e}^{\mathrm{i}\kappa} \qquad B = \frac{\zeta a + w}{d} \tag{55}$$

$$y = x - (2\zeta^2 + a^2 - k_0)t + y_0 \qquad d = \sqrt{a^2\zeta^2 - w^2} \tag{56}$$

with y_0 being a complex integration constant.

For ζ and d being real A is of modulus 1. If we then require that γ should be of modulus 1 the imaginary part of $2dy_0$ has to be a multiple of π and—due to property (2) of $\mathcal{S}$ in appendix B—we get two solutions

$$\gamma_\epsilon = \epsilon\mathcal{S}(\epsilon A, 2\,\mathrm{d}y) \qquad \epsilon = \pm 1 \tag{57}$$

with y defined by (56) where y_0 is now taken as real. For ζ real and d imaginary A is real. With y_0 assumed as real then—due to the property (3.2), see appendix B—it holds as well $|\gamma| = 1$.

Case 1. $N = 1$. For one real eigenvalue ζ_1 by (22) we get the solution

$$q^{[1]} = -\gamma_1(\gamma_1 a + 2\zeta_1)\exp[-\mathrm{i}k_0(x - (a^2 - k_0)t)] \tag{58}$$

with the convention that $\gamma_1, A_1, y_1, \ldots$ denote $\gamma, A, y, \ldots$ defined above with ζ replaced by ζ_1. y_0 is put equal to zero. Here $a > 0, k, \zeta_1$ are free real parameters. For $d_1^2 < 0$ (58) is a periodic solution. For $d_1^2 > 0$ the related intensity may be written as

$$\begin{aligned} |q^{[1]}|^2 &= |\gamma_1 a + 2\zeta_1|^2 \\ &= a^2 - 2k_0 - \frac{4d_1^2}{w_1 - \epsilon\zeta_1 a\cosh(2d_1 y_1)}. \end{aligned} \tag{59}$$

If we write

$$f_\pm \equiv \zeta_1 a \pm w_1 \qquad w_1 = \left(\zeta_1^2 + k_0/2\right) \tag{60}$$

solitons and periodic solutions are separated in the ζ_1–k_0 plane by the two parabolas $f_+ = 0$ and $f_- = 0$, see figure 1. For each point in the region $d_1^2 \equiv f_+ f_- > 0$ we get both a bright and a dark soliton depending on the sign factor ϵ. It can be seen, however, that the solution with ϵ being changed is the same as that with the sign of ζ_1 being changed. Thus we may take $\epsilon = +1$ without loss of generality. At the border $f_+ = 0$ or $f_- = 0$ one finds one solution of Lorentzian shape, the other of constant amplitude. Note that formula (59) does not contain solitons over vacuum because $d_1^2 = f_+ f_- \leqslant 0$ for $a^2 - 2k = 0$. In the limit $k = a^2/2$, $\zeta = a/2$, $\epsilon = +1$, however, one gets a Lorentzian pulse over vacuum.

Case 2. $N = 2$. Again we assume that either both the eigenvalues ζ_1, ζ_2 are real or one is complex conjugate to the other. The result of a two-fold Bäcklund transformation commonly for both these cases may be written as

$$q^{[2]} = \frac{K_1 a + 2K_3}{K_2^2} K_1 \exp[\mathrm{i}k_0(x - (a^2 - k_0)t] \tag{61}$$

$$K_1 = \zeta_1\gamma_1 - \zeta_2\gamma_2 \qquad K_2 = \zeta_1\gamma_2 - \zeta_2\gamma_1 \qquad K_3 = \zeta_1^2 - \zeta_2^2 \tag{62}$$

$$\gamma_j = \mathcal{S}(A_j, 2d_j y_j) \qquad j = 1, 2 \tag{63}$$

$$A_j = (-w_j + \mathrm{i}d_j)/\zeta_j a \tag{64}$$

$$d_j = \sqrt{\zeta_j^2 a^2 - w_j^2} \tag{65}$$

$$y_j = x - \left(2\zeta_j^2 + a^2 - k_0\right)t + y_{0j} \tag{66}$$

$$\text{(2a) } \zeta_1, \zeta_2, y_{01}, y_{02} \text{ real} \qquad \text{or} \qquad \text{(2b) } \zeta_2 = \zeta_1^* \quad y_{02} = y_{01}^*. \tag{67}$$

Consequently, it holds either (2a) $|\gamma_1| = |\gamma_2| = 1$ or (2b) $\gamma_2\gamma_1^* = 1$. Commonly for (2a) and (2b) the intensity may be written in the simple form

$$|q^{[2]}|^2 = |a + 2K_3/K_1|^2. \tag{68}$$

The solutions under (2b) coincide with the 'two-parametric solitons' of Mjølhus [3, 4]. We did not succeed in proving this analytically. But we checked numerically that $b(x,t)$ given by equations (46) and (47) in [4] coincides with our $q(x,t)$ up to some translation in x and t.

Due to the choice of real eigenvalues ζ_1, ζ_2 above formulae (61)–(68) may describe the collision of two bright solitons, figure 2(*a*), of two dark solitons, figure 2(*b*), or of one bright and one dark soliton, figure 2(*c*). Furthermore, with complex conjugate eigenvalues, we find a breather-type solution, see figure 2(*d*).

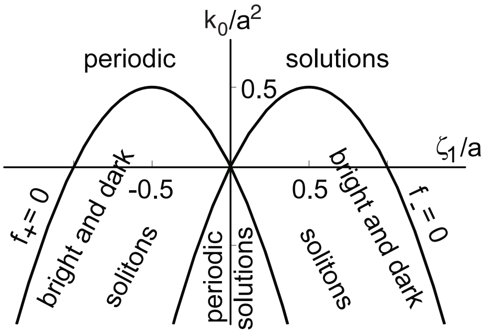

Figure 1. Regions for solitons and periodic solutions in the ζ_1–k_0 plane, where k_0 is the wave number of the seed solution and ζ_1 is the real parameter. The regions are separated by the two parabolas $f_+ = 0$ and $f_- = 0$, see equation (60).

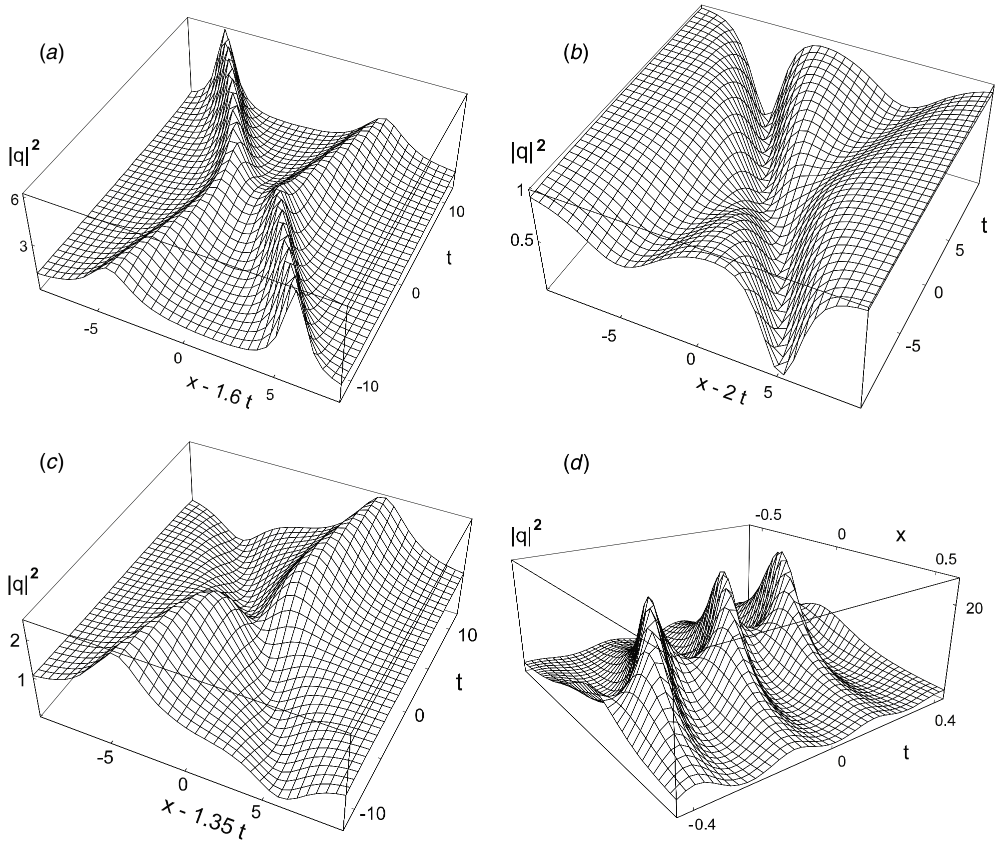

Figure 2. Four 3D plots of the two-soliton formula (68). (*a*) Collision of two bright solitons. $k_0 = -0.15$, $\zeta_1 = -1/3$, $\zeta_2 = 2/3$. (*b*) Collision of two dark solitons. $k_0 = -0.5$, $\zeta_1 = 1/3$, $\zeta_2 = -2/3$. (*c*) Collision of a bright soliton with a dark soliton. $k_0 = -0.1$, $\zeta_1 = -1/3$, $\zeta_2 = -2/3$. (*d*) Breather-type solution. $a = 1$, $k = -0.15$, $\zeta_1 = 1 + \mathrm{i}$, $\zeta_2 = 1 - \mathrm{i}$.

Case 3. $N = 2n$. Clearly, the general formulae of subsection 4.2 may be specified to a monochromatic wave as the seed function and the reduction $r = q^*$. For sake of shortness we will do it for even N only. Let all eigenvalues ζ_j, $j = 1, \ldots, 2n$, be either real or complex conjugate pairs, $\zeta_l = \zeta_k^*$, and take the integration constants y_{0j} as real or complex conjugate pairs, respectively. The functions γ_j and all auxiliary entities with subscripts j are given as before by (63)–(66), and the γ_j are fulfilling either $|\gamma_j| = 1$ or, pairwise, $\gamma_l\gamma_k^* = 1$. Then, corresponding to (51), we get

$$\beta_j = -\mathrm{i}\gamma_j \exp[\mathrm{i}k_0(-x + (a^2 - k_0)t)] \tag{69}$$

which has to be substituted into (32) and (33). Due to (A5), see appendix A, we get

$$p_0 = (-1)^{n-1} \frac{\mathcal{V}_{nn}\left(1; \zeta_j\gamma_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2; \zeta_j\gamma_j \middle| \zeta_j^2\right)} \tag{70}$$

$$p_1 = -\mathrm{i}\tilde{p}_1\, \mathrm{e}^{\mathrm{i}k_0(x-(a^2-k_0)t)} \qquad \tilde{p}_1 = \frac{\mathcal{V}_{n+1,n-1}\left(1; \zeta_j\gamma_j \middle| \zeta_j^2\right)}{\mathcal{V}_{nn}\left(\zeta_j^2; \zeta_j\gamma_j \middle| \zeta_j^2\right)} \tag{71}$$

and the $2n$-soliton solution is finally given as

$$q^{[2n]} = \frac{ap_0 + 2\tilde{p}_1}{p_0^*}\, \mathrm{e}^{\mathrm{i}k_0(x-(a^2-k_0)t)}. \tag{72}$$

Let us stay for a moment at this formula. Typically, N-soliton formulae are given as rational functions of polynomials of exponential functions. Their numerical evaluation in some large spacetime region is then made difficult because the exponential functions may become very large or very small. Also it is not easy to discuss the asymptotic behaviour. Here, however, up to a simple phase factor the formula is given in terms of functions $\gamma_j(x, t)$ with rather restricted regions of variability, cf appendix B. When the parameters are fixed such that periodic functions γ_j are excluded, we get a system of solitons and breathers with well-defined asymptotic directions of propagation. The asymptotic behaviour at any spacetime direction different from these specified ones then is simply that all γ become constants, and $q^{[2n]}$ becomes a monochromatic wave. Compared to the seed solution, q, frequency, wave number and amplitude of the asymptotic wave remain unchanged. Note, however, that for N odd the wave number changes sign and the asymptotic amplitude is changed.

Three examples of the collision of eight solitons are depicted in figure 3. From the middle part, figure 3(*b*), we see that the chain of eight solitons, alternately bright and dark, can result in a steepening of intensity by one order of magnitude in the collision centre.

5. Stability

The stability of DNLS solitons has been investigated by Mjølhus and co-workers [3, 35]. Their main results may be summarized as follows: one-soliton states are stable except when they lie on the border of their region of existence as indicated in figure 1, i.e. when they are of algebraic type. Breather-type solitons with two complex-conjugate eigenvalues $\zeta_2 = \zeta_1^*$ are stable as well. But in the limit $\zeta_2 = \zeta_1 =$ real, when there is only one degenerate eigenvalue, stability is lost.

We will add a remark concerning the asymptotic state. Clearly, a necessary condition for a soliton over a finite background to be stable is that its asymptotic states for $x \to \pm\infty$ must be stable. These states are monochromatic waves with equal amplitudes left and right but with some difference in their phases. The stability of DNLS waves of the form (49) has been investigated in the literature by two ways.

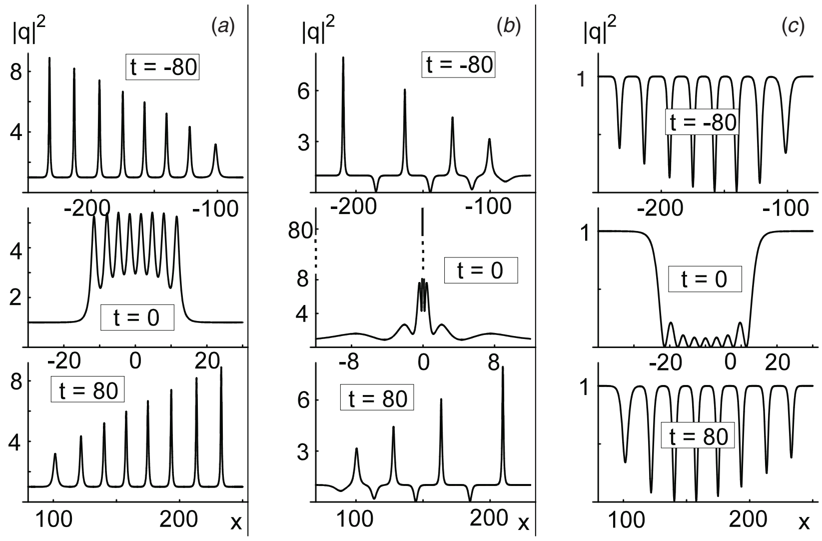

Figure 3. Eight-soliton solutions. (*a*) Eight bright solitons. $k_0 = -0.2$, $\zeta_j^a = (-0.3, 0.45, -0.55, 0.63, -0.7, 0.77, -0.84, 0.9)$. (*b*) Four bright and four dark solitons. $k = -0.5$, ζ_j^b equidistant, $\zeta_1 = 0.19$, $\zeta_8 = 0.82$. (*c*) Eight dark solitons. $k_0 = -0.2$, $\zeta_j^c = -\zeta_j^a$.

(i) *Linear stability* [20]. The DNLS equation (1) is taken for the wave $q = a \exp[\mathrm{i}k(x - (a^2 - k)t)]$ plus some perturbation and is linearized with respect to this perturbation. The condition that there are no exponentially growing modes is then found to be

$$k < a^2/2. \tag{73}$$

(ii) *Stability against finite harmonic perturbations* [2, 36]. The ansatz

$$q = a(x,t)\,\mathrm{e}^{\mathrm{i}\theta(x,t)} \qquad \theta_x = k \qquad \theta_t = -\omega \tag{74}$$

is substituted in (1). Together with $k_t = \omega_x$ and by neglecting the second derivative of a, one finds a first-order quasi-linear system of partial differential equations for $a^2(x,t)$ and $k(x,t)$. The stability criterion then is that this system has to be of hyperbolic type, and this is fulfilled just under the same condition (73).

Applied to the seed solution (49) the condition (73) reads $k_0 < a^2/2$. From figure 1 then we may see that solitons can be generated only from stable seed solutions. We remember that the wave number changes sign under the Bäcklund transformation, cf (58), and that the asymptotic amplitude of the transformed solution becomes $\sqrt{a^2 - 2k_0}$. Thus the condition of asymptotic stability reads $-k_0 < a^2/2 - k_0$ and is trivially fulfilled. That is, the asymptotic states of all one-soliton solutions are stable. A numerical test whether or not DNLS solitons remain stable in the parent system, from which the DNLS equation has been derived, has been made in [7] for the example of magnetohydrodynamic waves propagating in a high-β plasma. Large-amplitude DNLS solitons, introduced as initial conditions for the fully nonlinear Hall-MHD system, exhibited a remarkable degree of stability, with some differences in favour of dark solitons.

6. Summary and conclusions

By use of a rather simple and elementary approach to Darboux transformations we derived formulae for N-soliton solutions both over vacuum and over a finite background. These formulae are written in terms of Vandermonde-like determinants. In order to estimate the advantage of these determinants and their structural properties the reader should have in mind that for one of the three eight-soliton solutions depicted in figure 3 one has to evaluate determinants of 16×16 matrices with $16! \sim 2 \times 10^{13}$ terms. Due to the *reduction formula* (A2), appendix A, for computing $\mathcal{V}_{44}$, this huge number is reduced to $\binom{16}{8} = 12\,870$ terms so that the problem becomes tractable. Also, it proves to be advantageous to introduce an elementary auxiliary function, $\mathcal{S}(A, z)$, called the *seahorse function* which approaches finite values A or $1/A$ for $|x| \to +\infty$ or $-\infty$, respectively.

Imai [26] has developed a Darboux/Bäcklund transformation technique which is similar to ours. His treatment is more general insofar as he deals with some class of compatible linear systems. Our approach, however, seems to be simpler, and our results are more explicit and more appropriate for numerical evaluation. His classification of DNLS solitons is incomplete because only solutions generated by multiple Bäcklund transformations starting from the vacuum are considered. As we have pointed out in subsection 5.2 only a special type of monochromatic waves can be generated in this way and, consequently, only a submanifold of the whole soliton hierarchy can be generated when starting from the zero solution.

Acknowledgments

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Appendix A. Vandermonde-like determinants

Vandermonde-like determinants are defined as follows [25]:

$$\mathcal{V}_{MN}(a_r; b_r|x_r) := \begin{vmatrix} a_1 & a_1x_1 & \cdots & a_1x_1^{M-1} & b_1 & b_1x_1 & \cdots & b_1x_1^{N-1} \\ a_2 & a_2x_2 & \cdots & a_2x_2^{M-1} & b_2 & b_2x_2 & \cdots & b_2x_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{M+N} & a_{M+N}x_{M+N} & \cdots & a_{M+N}x_{M+N}^{M-1} & b_{M+N} & b_{M+N}x_{M+N} & \cdots & b_{M+N}x_{M+N}^{N-1} \end{vmatrix} \quad \text{(A1)}$$

where $r = 1, 2, \ldots, (M + N)$. These determinants have several remarkable structural properties listed in [25]. In particular, any Vandermonde-like determinant $\mathcal{V}_{MN}$ can be expressed as a sum over binary products of genuine Vandermonde determinants $\mathcal{V}_N$. This is done by the *reduction formula*

$$\mathcal{V}_{MN}(a_r; b_r|x_r) = \sum_P \varepsilon_P \prod_{j=1}^{M} a_{s(j)} \prod_{k=M+1}^{M+N} b_{s(k)} \mathcal{V}_M(x_{s(1)} \ldots x_{s(M)}) \mathcal{V}_N(x_{s(N+1)} \ldots x_{s(M+N)}). \quad \text{(A2)}$$

The sum goes over all permutations $P = (s(1), \ldots, s(M+N))$ of $(1, 2, \ldots, M+N)$ such that $s(i) < s(j)$ for $i < j \leqslant N$ as well as for $N < i < j$. Permutations of such a type are called *shuffles* [37]. $\varepsilon_P = +1$ for P even or -1 for P odd. The (genuine) Vandermonde determinants $\mathcal{V}_N$ are defined as

$$\mathcal{V}_N(x_1, \ldots, x_N) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{vmatrix}. \quad \text{(A3)}$$

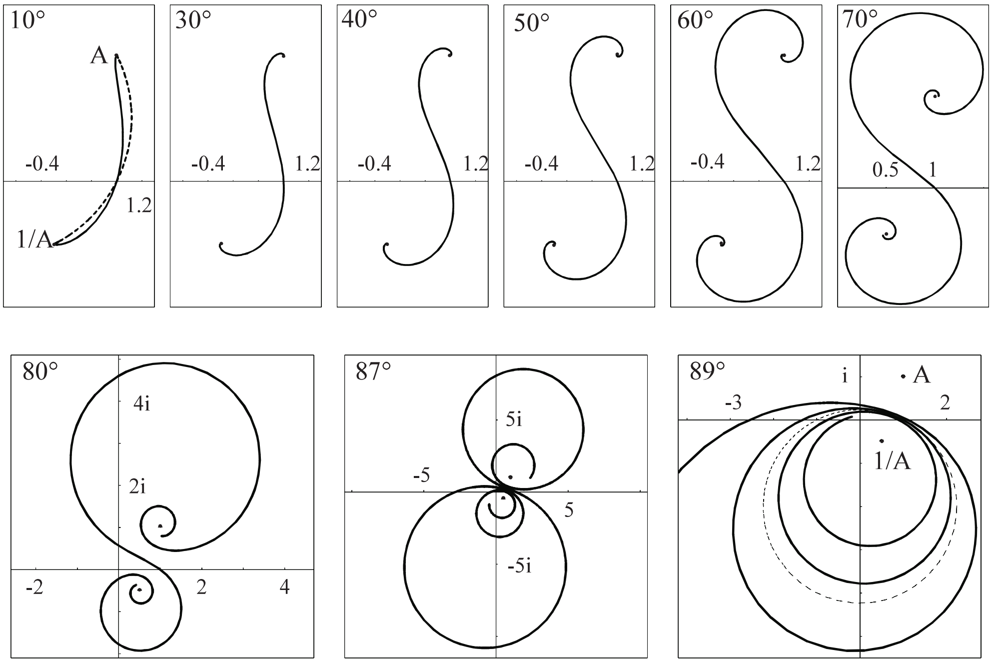

Figure 4. The seahorse function. For some straight lines in the complex z-plane the images of the mapping $z \to \mathcal{S}(A, z)$, $A = 1 + \mathrm{i}$, are depicted. In each of the nine parts of the figure the angle of declination against the real z-axis is indicated.

It is well known and can be easily checked directly that $\mathcal{V}_N$ can be written as a product of differences,

$$\mathcal{V}_N(x_1, \ldots, x_N) = \prod_{i>j}(x_i - x_j). \tag{A4}$$

From the definition (A1) it is obvious that a common factor of all b_r could be extracted,

$$\mathcal{V}_{MN}(a_r; b_r f|x_r) = f^N \mathcal{V}_{MN}(a_r; b_r|x_r). \tag{A5}$$

Appendix B. The seahorse function

Let us define three functions $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2$,

$$\mathcal{S}(A, z) \equiv \frac{A\exp(z) + 1}{\exp(z) + A} \tag{B1}$$

$$\mathcal{S}_1(B, z) \equiv \frac{\coth(z/2) + \mathrm{i}B}{\coth(z/2) - \mathrm{i}B} \tag{B2}$$

$$\mathcal{S}_2(\kappa, z) \equiv \frac{\cosh[(z + \mathrm{i}\kappa)/2]}{\cosh[(z - \mathrm{i}\kappa)/2]}. \tag{B3}$$

Then connections between the parameters A, B, κ may be postulated,

$$A = \exp[\mathrm{i}\kappa] \qquad B = \mathrm{i}\frac{1 - A}{1 + A} = \tan(\kappa/2). \tag{B4}$$

such that in any case we get the same function in z but with redefinition of the respective parameter.

$\mathcal{S}(A, z)$ solves the Riccati equation

$$\partial_z \mathcal{S} = -\frac{A}{A^2 - 1}(1 + \mathcal{S}^2) + \frac{A^2 + 1}{A^2 - 1}\mathcal{S}. \tag{B5}$$

We list some properties of $\mathcal{S}(A, z)$. With respect to our application we are interested, in particular, in the images of straight lines.

(1) $\lim_{\mathrm{Re}\, z \to \infty} \mathcal{S}(A, z) = A$, $\lim_{\mathrm{Re}\, z \to -\infty} \mathcal{S}(A, z) = 1/A$

(2) $\mathcal{S}(A, z + \mathrm{i}\pi) = -\mathcal{S}(-A, z)$.

(3.1) For $|A| = 1$ and real z as well as

(3.2) for real A and imaginary z it holds $|\mathcal{S}(A, z)| = 1$.

(4) The image of the real axis in the z-plane is an arc of a circle going from $1/A$ to A.

(5) The image of a piece of a straight line of length 2π parallel to the imaginary axis is a circle.

(6) The image of any straight line not parallel to one of the axes is an S-shaped curve approaching $1/A$ and A on spirals, see figure 4 for an example. With a little imagination the reader may see there a seahorse.

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